Optical frequency measurements and frequency combs

Y. Le Coq

LNE-SYRTE – Observatoire de Paris, CNRS, UPMC, Paris, France
yann.lecoq@obspm.fr

OCS2018 Tutorial, Gressoney, Italy September 2018
General concepts about noise characterization
A signal with some noise: \[ s(t) = s_0 + x(t) \]

Three basic concepts to characterize this signal:

- **Power Spectral Density of Noise**: \( S_x(f) \)
  - Typ. fluctuations of \( s(t) \) at a given Fourier frequency \( f \)

- **Signal stability**: \( \sigma_x(\tau) \)
  - Fluctuation of \( s(t) \) at a characteristic time \( \tau \)

- **Accuracy**
  - How much do we know \( <s(t)> \) converges toward what we want i.e. \( s_0 \)
  - Systematic error from different parameters evaluation (all you can think about... and more)
  - NB: a good stability makes such evaluation easier!
Power Spectral Density (PSS)

If $x(t)$ is a stochastic physical quantity, of unit [unité]

One defines

$$x_T(t) \triangleq \begin{cases} x(t) & \text{pour} \quad |t| \leq T \\ 0 & \text{pour} \quad |t| > T \end{cases}$$

(finite support, hence its FT exists...)

The PSD of $x$ is:

$$S_x(f) \triangleq \lim_{T \to \infty} \frac{\mathbb{E} \left[ |X_T(f)|^2 \right]}{2T}$$

in [unit$^2$.Hz$^{-1}$]

NOTE: alternatively, one can also define PSD as:

$$S_x(f) = \int_{-\infty}^{+\infty} R_{xx}(\tau)e^{-2\pi if\tau} \, d\tau$$

with $R_{xx}(\tau) \triangleq \mathbb{E} [x(t) \cdot x(t + \tau)]$

(from Wiener-Khintchine theorem, these 2 definitions are equivalent)

For $x(t)$ real, $S_x(f)$ is even, therefore, with no loss on information, one uses the one-sided PSD

$$S_{xOS}(f) = S_s(f) + S_x(-f) = 2S_x(f) \quad \text{pour} \quad f > 0$$

et $S_{xOS}(0) = S_x(0)$

This is normally the quantity that is used (eg.: FFT analyzer)!

Properties:

$$S_{\alpha x} = \alpha^2 S_x \quad S_{d \, x/dt}(f) = f^2 S_x(f) \quad S_{x_1+x_2} = S_{x_1} + S_{x_2} \quad \text{For} \quad x_1, x_2 \text{indep.}$$
A PSD estimator: averaged periodogram

If \( x(t) \) is a stochastic physical quantity sampled in \( x_n \) every \( \Delta t \) using ergodicity and stationnarity of the noise process:

\[
S_x(f_i) \text{ avec } \Delta f_i = 1/(2^n \Delta t)
\]

Notes:
- Generally necessary to use a windowing function (ex.: Hanning) before the FFT to limit spectral overlap
- If a window function is used, one can overlap blocks (Welsh method)

Très bien pour représentation log-lin ou lin-lin,
MAIS pas très pratique si représentation log-log sur plusieurs décades :
→ grande résolution (nombre de pts) sur la dernière décade, très faible sur la première
Importance of averaging

\[ |\text{FFT}|^2, \text{512 points, 1/f noise} \]

Credit: www.dsprelated.com

Note: when plotting in log-log over several decades of Fourier frequencies, the resolution is mediocre on the left, and too high on the right...
An exemple of PSD algorithm

We have $N(>1024\times1000\sim1M\ \text{here})$ samples of a signal $\text{DataA}[n]$, sampled at 2MSPS. We want the PSD over $>1$ decade of Fourier frequencies (log-log).

```python
SAMPLING_FREQUENCY = 2000000.0  # 2MSPS used here
nfft=1024  #1024 points of fft in each decade
decades = 4  # for Fourier frequencies from 100Hz to 1MHz (with still some info <100Hz from the lowest decade)
window = scipy.signal.windows.get_window('blackmanharris', nfft)  # blackmanharris windowing is used here

def PSDcore(dataA, nfft, decades, window):
    # single sided power spectral density
    scaling = 2.0 / (SAMPLING_FREQUENCY * (window * window).sum())
    psdA = []

    for decade in range(decades):  # ie 0,1,2,3 here
        if decade > 0:
            dataA = scipy.signal.decimate(dataA, 10)
        fftA = rfft(window * np.reshape(dataA, [-1, nfft]))

        A = np.mean(fftA * fftA.conj(), axis=0).real * scaling * 10**decade
        relative_f = rfftfreq(nfft, 0.5)
        mask = np.ones(f.shape, dtype=bool)
        if decade != decades - 1:
            mask = mask & (relative_f > 0.08)
        if decade > 0:
            mask = mask & (relative_f <= 0.8)
        psdA.append(A[mask])
    psdA = np.concatenate(psdA[::-1])

    return psdA
```

NB.: if data arrives continuously by blocks of $N$ samples, averaging the results from PSDcore(), the estimator of PSD is improving over time.
An exemple of PSD algorithm

A typical result of such algorithm

![Er-doped fiber laser RIN](image)
Allan deviation/variance

Statistical tool introduced by David Allan to characterize the fluctuations of the frequency of oscillators. Nevertheless, can be applied to any kind of measurement...

\[ s(t) = s_0 + x(t) \quad [\text{unit}] \]

Definition:

\[ \sigma_x^2(\tau) = \frac{1}{2} < (x_{n+1} - x_n)^2 > \quad (\text{Variance, unit}^2) \]

\[ \sigma_x(\tau) = \sqrt{\frac{1}{2} < (x_{n+1} - x_n)^2 >} \quad (\text{Deviation, unit}) \]

Utility:

- (much) faster to estimate than the PSD for “long“ characteristic times \( \tau \)
  (ie low Fourier frequencies)

- defined when the PSD may be isn't (eg. drift)

- defined when the true average \(< s(t) >\) maybe isn't (eg. drift or LF colored noise)

- gives the “useful” time of measurement:
  If the AVAR is cte or increases after a time \( > \tau \)
  averaging for longer than \( \tau \) is useless, or even bad (more noise when you average more...)!
One can actually calculate the AVAR from the PSD:

$$\sigma_x^2(\tau) = 2 \int_0^\infty S_x(f) \frac{\sin^4(\pi \tau f)}{(\pi f)^2}$$

The other way around is not true... except if you make hypothesis on the noise, typically: $S_x(f) \propto f^\alpha$

<table>
<thead>
<tr>
<th>Noise type</th>
<th>$S_x(f)$</th>
<th>$\sigma_x^2(\tau)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linear drift</td>
<td>$-$</td>
<td>$K\tau^2$</td>
</tr>
<tr>
<td>Random walk</td>
<td>$h_{-2}f^{-2}$</td>
<td>$h_{-2}A\tau^1$</td>
</tr>
<tr>
<td>Flicker (pink)</td>
<td>$h_{-1}f^{-1}$</td>
<td>$h_{-1}B\tau^0$</td>
</tr>
<tr>
<td>white</td>
<td>$h_0f^0$</td>
<td>$h_0C\tau^{-1}$</td>
</tr>
<tr>
<td>blue</td>
<td>$h_1f^1$</td>
<td>$h_1D\tau^{-2}$</td>
</tr>
<tr>
<td>violet</td>
<td>$h_2f^2$</td>
<td>$h_2E\tau^{-2}$</td>
</tr>
</tbody>
</table>

With:

- $K = \Delta^2/2$ for $\Delta$ the linear drift
- $A = 2\pi^2/3$
- $B = 2\ln(2)$
- $C = 1/2$
- $D = [1.038 + 3\ln(2\pi f_h\tau)]/4\pi^2$
- $E = 3f_h/4\pi^2$

With $f_h$ a high-frequency cut-off, that allows convergence...

Note: table largely used in time/frequency when taking about the frequency fluctuations of an oscillator, but in works for any noisy measurement in general.
The Allan variance allows identification of the type of noise (power law)

It doesn't distinguish between noises in $f^1$ and $f^2$. Other 2 sample variances exist for that (eg. MVAR)
AVAR algorithm

Exemple of software packages

Alavar www.alamath.com
Stable32 www.wriley.com

AllanTools (python module)

![Graph showing Allan deviation over time](image)

Exemple of python code

```python
def adev(x, tau, sampl=1.0):
    """Allan deviation""
    x = np.asarray(x)
    tau = np.asarray(tau)

    # allocate output vectors
    adev = np.zeros(tau.size)
    dadev = np.zeros(tau.size)

    # samples
    n = x.size
    # partitioning
    p = np.floor(tau * sampl).astype(int)

    for i, m in enumerate(p):
        d = x[n:n + m].reshape(-1, m)
        y = np.mean(d, axis=1)
        adev[i] = np.sqrt(0.5 * np.mean(np.diff(y)**2))
        dadev[i] = adev[i] / np.sqrt(y.size)

    return adev, dadev
```

Note: error bars in $\sigma(\tau) / \sqrt{N}$ for large $N$ (for small $N$, a bit different, depending on the kind of noise...
Case of a sinusoidal oscillator
Toolbox for an oscillator…

Oscillator: \( s(t) = A_0 \cdot [1 + \alpha(t)] \cdot \cos[2\pi \nu_0 t + \phi_0 + \phi(t)] \)

\( \nu_0 \): GHz range: microwave oscillator

100’s THz range: optical oscillator

\( \alpha(t) \): amplitude noise

(generally neglected wrt phase noise, bounded)

\( \phi(t) \): phase noise

(not bounded, always divergent)

\( \nu(t) = \nu_0 + d\phi/dt(t) \)

instantaneous frequency

PSD phase noise:
\( S_\phi(f) \)

PSD ampl. noise (RIN):
\( S_\alpha(f) \)

PSD signal:
(spectrum analyz. \( S_s(f) \))

+ stabilité de fréquence (en Hz): \( \sigma_\nu(\tau) \)
+ stabilité de fréquence relative: \( \sigma_y(\tau) \) où \( y = \nu/\nu_0 \)
+ stabilité d'amplitude (relative): \( \sigma_\alpha/A_0(\tau) \)
+ frequency accuracy
\[ s(t) = A_0 \cdot [1 + \alpha(t)] \cdot \cos[2\pi\nu_0 t + \phi_0 + \phi(t)] \]

**Phase noise:** \( \phi(t) [\text{rad}] \rightarrow S\phi(f) [\text{rad}^2/\text{Hz}] \) (one-sided PSD)

ou, plus usuellement [dB rad^2/Hz] ou [dB rad/\sqrt{Hz}]

deux noms pour la même chose, ie 10.log10(S\phi(f))

By convention, one uses also \( L_c(f) = 10.\log_{10}(S\phi(f)) - 3\text{dB} \) [dBc/Hz]

**Amplitude noise:** \( \alpha(t) [\text{relave, unitless}] \rightarrow S\alpha(f) [\text{Hz}^{-1}] \) (one-sided PSD)

Usually, one also uses for phase noise \( L_c(f) = 10.\log_{10}(S\alpha(f)) - 3\text{dB} \) [dBc/Hz]

Both noises can be expressed in \( L_c(f) \) [dBc/Hz] because of a historic relation with \( S_s(\nu'-\nu) \) (sideband/carrier ratio on a spectrum analyzer BUT only true under very restrictive conditions of low noise)

IEEE Standard Definitions of Physical Quantities for Fundamental Frequency and Time Metrology—Random Instabilities ; Standard 1139, revision 1999
Two more concepts...

\[ s(t) = A_0 \cdot [1 + \alpha(t)] \cdot \cos[2\pi \nu_0 t + \phi_0 + \phi(t)] \]

**Oscillator linewidth:**

Many different definitions are possible and used.
La plus raisonnable, pour bruit d'amplitude négligeable vs. bruit de phase:

Linewidth \( \Delta \nu \) defined as:
\[
\int_{\Delta \nu / 2}^{\infty} S_\phi(f) \cdot df = \frac{2}{\pi}
\]

(corresponds to FWHM spectrum linewidth
for white phase noise \( S_\phi(f) = S_\nu / f^2 = \text{cte} / f^2 \))

**Timing jitter**

\[
S_{\Delta t}(f) = \frac{S_\phi(f)}{(2\pi \nu_0)^2}
\]  
integrated jitter; = \[
\int_{f_{\text{min}}}^{f_{\text{max}}} S_{\Delta t}(f) \, df
\]

Particularly useful when comparing oscillators which oscillates at different carrier frequencies...
A word of advice

Talking about a “very low phase noise oscillator” – without context – is meaningless.

The phase noise level is meaningless only wrt the carrier frequency !!!

Exemple.:

-170dBc/Hz at 10kHz from a de 10GHz carrier is a worl record

-170dBc/Hz at 10kHz from a 100MHz is just normal for a good commercial quarz oscillator costing a few k€/pièce (ex. Rakon LNO100)
Frequency division: effect on phase noise

\[\begin{align*}
\text{fc [Hz]} & \rightarrow \text{fc/N [Hz]} \quad \text{then} \quad \Delta \phi & \rightarrow \Delta \phi/N \text{ [rad]} \\
S_\phi(f) \text{ [dBc/Hz]} & \rightarrow S_\phi(f) - 20 \cdot \log_{10}(N) \text{ [dBc/Hz]}
\end{align*}\]

Exemple : division by 2

Large reduction of phase noise for large N…
Frequency division: effect on phase noise

fc [Hz] → fc/N [Hz]  alors Δϕ → Δϕ/N [rad]

S_ϕ(f) [dBc/Hz] → S_ϕ(f) – 20.log10(N) [dBc/Hz]
Frequency division: effect on phase noise

\[ \text{fc [Hz]} \rightarrow \text{fc/N [Hz]} \quad \text{alors} \quad \Delta \phi \rightarrow \Delta \phi/N \quad \text{[rad]} \]

\[ S_{\phi}(f) \quad \text{[dBc/Hz]} \rightarrow \quad S_{\phi}(f) - 20 \cdot \log_{10}(N) \quad \text{[dBc/Hz]} \]

Exemple : division by 2

Dephasing \( \Delta \phi \)

\( \rightarrow \) shift in time \( \Delta t = \Delta \phi/(2\pi fc) \)
Division de fréquence : effet sur le bruit de phase

\[ fc \text{ [Hz]} \rightarrow fc/N \text{ [Hz]} \text{ then } \Delta f \rightarrow \Delta f/N \text{ [rad]} \]

\[ S_\phi (f) \text{ [dBc/Hz]} \rightarrow S_\phi (f) - 20 \cdot \log_{10}(N) \text{ [dBc/Hz]} \]

Exemple : division by 2

Time shift \( \Delta t = \Delta \phi / (2\pi fc) \) on \( fc/2 \) signal → dephasing \( \Delta \phi' = \Delta \phi / 2 \) on \( fc/2 \)
Case of a femto-second laser
Output of a mode-locked Laser

\[ E(t) \]

\[ T_R = \frac{1}{f_{\text{rep}}} \]

\( f_{\text{rep}} \) is the pulses repetition rate

A priori: \( >10^5 \) modes, each one a sine-wave-like oscillator!

A lot of possible degrees of freedom, each potentially noisy!!!

BUT: mode-lock laser = controlled dispersion in the laser cavity (GVD=0) otherwise the pulses don't “hang together” and break apart
Output of a mode-locked Laser

\[ T_R = \frac{1}{f_{\text{rep}}} \quad \text{where } f_{\text{rep}} \text{ is the pulses repetition rate} \]

Pulses inside the cavity with \( v_g \neq v_\phi \) → the pulses “hang together”, so there must be no group velocity dispersion (GVD), but there can be phase velocity dispersion!

Zero GVD means:

\[ \frac{\partial k(\omega)}{\partial \omega} = \frac{1}{v_g} = \text{const.} \]

\[ \frac{\partial^m k(\omega)}{\partial \omega^m} = 0, \forall m > 1 \]

This is satisfied for:

\[ k(\omega) = n(\omega)\omega/c = k_0 + \omega/v_g \]

Which corresponds to:

\[ n(\omega) = c \left( \frac{1}{v_g} + \frac{k_0}{\omega} \right) \]

(on average over 1 round trip of the cavity)
If we plug \( n = c \left( \frac{1}{v_g} + \frac{k}{\omega} \right) \) into the freq. of the FP cavity modes \( \omega_N = N \frac{2\pi}{nL} \) and solve for the frequencies, we get: 

\[
\omega_N = N \frac{2\pi v_g}{L} - k_0 v_g
\]

This is the celebrated formula \( \nu_N = N \times f_{\text{rep}} + f_0 \)

Note: usually, we choose \( N \) such that \( 0 < f_0 < f_{\text{rep}}/2 \)

The comb is a « ruler » in optical frequency domain, with one mode every \( f_{\text{rep}} \)

There is a **fixed phase relationship** between the different modes!

→ phase noise of \( f_{\text{rep}} \) and of \( f_0 \) (plus ~pulse amplitude noise) is sufficient to characterize the noise properties of the femtosecond laser...

(note consequence on Schalow-Townes limit for mode \( \nu_N << \text{cw with same power...} \))
Output of a mode-locked Laser

- In time domain (Fourier transform), this corresponds to:
  
  \[ f_0 = f_r \times \frac{\Delta \phi}{2\pi} \]

  ![Diagram showing carrier envelop phase shift](image)

  Carrier envelop phase shift \( \Delta \phi \) → 2\( \Delta \phi \)

- After getting out of the cavity, the pulse can experience further dispersion (\( \text{GVD} \neq 0 \): blue and red part propagate at different speed)
  → chirped pulse in time domain // spectral phase in frequency domain

  ![Diagram showing chirped pulse and frequency chirp](image)

  Quadratic phase dependance = frequency chirp
  Linear phase dependance = time shift

  \[ v_m = mf_{\text{rep}} + f_0 \]
Optical frequency comb
Beat note between two lasers

\[ E_2^{(0)} \cos(\omega_2 t - k_2 z) \]

Laser 2

\[ E_1^{(0)} \cos(\omega_1 t - k_1 z) \]

Laser 1

\[ |\omega_1 - \omega_2| \ll \omega_1, \omega_2 \]

Beam splitter

Electric field: \[ E(t) = E_1 \cos(\omega_1 t - \phi_1) + E_2 \cos(\omega_2 t - \phi_2) \]

Poynting vector:
\[
\Pi(t) \propto |E(t)|^2 = E_1^2 \cos^2(\omega_1 t - \phi_1) + E_2^2 \cos^2(\omega_2 t - \phi_2) + 2E_1E_2 \cos(\omega_1 t - \phi_1) \cos(\omega_2 t - \phi_2)
\]
\[
= \frac{E_1^2}{2} \left[ 1 + \cos(2\omega_1 t - 2\phi_1) \right] + \frac{E_2^2}{2} \left[ 1 + \cos(2\omega_2 t - 2\phi_2) \right]
\]
\[
+ E_1E_2 \left[ \cos((\omega_1 - \omega_2)t - (\phi_1 - \phi_2)) + \cos((\omega_1 + \omega_2)t - (\phi_1 + \phi_2)) \right]
\]

Average optical power over detector time constant:
\[ P_{opt}(t) \propto E_1^2 + E_2^2 + 2E_1E_2 \cos((\omega_1 - \omega_2)t - (\phi_1 - \phi_2)) \]
\[ |\omega_1 - \omega_2| \ll 1/\tau_{det} \ll \omega_1, \omega_2 \]

Beat note between the 2 lasers at frequency: \[ |\omega_1 - \omega_2| \]

Current through the load:
\[ I(t) = \eta P_{opt}(t) \]

Beat note RF power:
\[ P_{RF} = < R_{load} \times I^2(t) > \propto R_{load} \times \eta^2 |E_1|^2 |E_2|^2 \sim R_{load} \times \eta^2 P_{1opt} P_{2opt} \]
Fast photodiodes and electronics are typically limited to <100 GHz.

100 GHz = 0.1 THz is a tiny fraction UV/VIS/IR range (~1000 THz).

→ in practice beat note measurements are **narrow band** at the scale of the UV/VIS/IR spectrum.

Detecting a beat note is similar to using a mixer in the RF or microwave domain. However, a mixer usually gives access to both \( \omega_1 - \omega_2 \) and \( \omega_1 + \omega_2 \).

Other methods to “move” in the optical frequency domain:

- Second Harmonic Generation, Sum and Difference Frequency Generation.
- Generally **narrow band** (crystals, coatings, geometry,… specific to a narrow wavelength range).

No straightforward way to link optical frequency to microwave frequency.
Harmonic frequency chain: Ensemble of subsystems each allowing a frequency multiplication by a factor 2 or 3.
- Complexity.
- Inconvenient frequencies (THz, mid-IR).
- 3 or 4 chains in the world at PTB, SYRTE, NRC, JILA.
- Continuous operation <3h.

The setup has to be redesigned if one changes the frequency to be measured.

H. Schnatz et al. PRL 76, 18 (1996)
Optical Frequency Measurements with Optical Combs

- Harmonic chain replaced with: 1 Laser

\[ \nu_N = Nf_{\text{rep}} + f_0 \quad \text{N large!} \]

- Extremely simple and cost effective in comparison.
- No inconvenient frequency.
- Measurement of several frequencies at the same time.
- Continuous operation for weeks.
- Covers the visible-near IR range.
- Commercially available systems.

Nobel Lecture: Defining and measuring optical frequencies
J. L. Hall, Rev. Mod. Phys. 78, 1279 (2006)

Nobel Lecture: Passion For Precision
Requirements for mode-locked operation

- Gain medium with broad gain curve to support a large number of modes (the largest known is Ti:Sapphire by far, but many other exists and researchers are working on making and characterizing some more…)

- Controlled dispersion to allow short pulses to remain short through round trip propagation in the cavity (glass is GVD > 0 in visible spectrum)
  - Prism pairs to compensate normal dispersion
  - Combining gratings and prism for shortest pulses (historical)
  - (doubly) chirped mirrors

*(Large oscillations of GDD otherwise, due to Gire-Tournois multiple interf.*)

*(Note GDD is expressed in ps^2, which is really ps/PHz…)*
Requirements for mode-locked operation

- Mechanism for (passive) mode-locking: Coupling between longitudinal modes so that they oscillate with a well defined phase with respect to each other. Any intra-cavity effect which will make high instantaneous intensity (pulsed) operation favorable will do. Examples:
  - Saturable absorber (often quite slow, carbon nanotubes?),
  - Non-linear polarisation rotation,
  - Non-linear fiber loop mittor
  - Kerr-lens effect (very fast)

- Startup by noise, more or less easily (self start, kick start, also synchronous pumping)
Example of a femtosecond Ti:Sa laser

- Gain medium is a Titanium doped sapphire crystal.
- Kerr-lens effect is making pulsed operation favorable.
- Pumped with ~6 W of laser light at 532 nm.
- Cavity made of 6 chirped mirrors.
- Intra cavity wedge for coarse tuning $f_0$.
- Fast and slow piezo actuators.
- Fast pump power control through an AOM.
- 30 fs pulses at 850 nm. Repetition rate: ~760 MHz.
- ~500 mW output power.
Example of a Er-doped fiber laser with non lin. pol. rotation

- Gain medium is a Er-doped fiber.
- Pumped with diode lasers at 980 nm.
- Double wedge for $f_0$ coarse tuning without too much $f_{\text{rep}}$ coupling: when translated total glass thickness is identical and n-index very close → group delay is conserved
  → BUT, $dn/d\lambda$ different → group delay dispersion is changed, hence $f_0$ too
- Actuation is possible via PZT (mainly frep)
- Pump power

\[ \begin{align*}
  n_1 &= n_0 + \frac{dn_1}{d\lambda}(\lambda-\lambda_0) \\
  n_2 &= n_0 + \frac{dn_2}{d\lambda}(\lambda-\lambda_0)
\end{align*} \]

Change of polarization along the fiber depends on the intensity of the light

- Non-linear polarization rotation in the fiber is making pulsed operation favorable when $\lambda$ plates are adjusted accordingly
Example of a Er-doped fiber laser with “figure 8” topology

- Gain medium is a Er-doped fiber.
- Pumped with diode lasers at 980 nm.

- Very robust, no tuning at all for mode locking (but usually low power), full PM possible

- Possible to add an EOM for feed back on $f_{\text{rep}}$

Figure from Encyclopedia of Laser physics (www.rp-photonics.com)
Measuring the comb parameters $f_R$ and $f_0$

- Measurement of $f_R$: detect the pulse train with a fast photodiode
  - Note: gives directly access to harmonics of $f_R$

- Measurement of $f_0$ with an octave spanning comb: The self-referencing method
  - Note: many modes contribute to the signal at $f_0$ (requires phase matching condition between each couple freq beats as red-doubled and blue part of the pulse arrive at the same time on the detector)

$$f(k) = f_0 + k \times f_r$$

$$f(2k) = f_0 + 2k \times f_r$$
Generation of an octave spanning optical frequency comb

- Short input pulses with high peak power
- Small core: high peak intensity
- Pulses remain short through propagation
- Large non-linear effects

- Time domain: Self phase modulation, cross-phase modulation, stimulated Raman emission,…
- Frequency domain: Four wave mixing

- Coherent spectral broadening
  - Note: pulses are broadened ➔ Spectral coherence but no Fourier limited pulses

Photonic crystal fiber

Core: 2µm

Dispersion management allows propagation of short pulses
Octave spanning optical frequency combs

- Ti:Sa, Er-doped fiber broadened with photonic crystal fiber
  

- Ultra broadband Ti:Sa
  

- Toroidal micro resonators
  
  P. Del'Haye et al., arxiv 0912.4890v1 (2009)
Phase locking and stuff...
Phase-locking (a comb or anything else…)

Noise sources

Physical system
Vin → ΔF

Physical actuator ALWAYS gets mad at high enough Fourier frequencies

G^[dB(ΔF/V)]

Bode diagram

Fo urier frequency

RF/μ-wave frequency to lock: F

(Mixer as a phase detector)

Slope near 0 V = Kd in [V/rad]

Phase detection

Loop filter (transfer function to be designed...)

Measure φ, act on F (via Vin) → equivalent to a pure integrator

G [dB]

Equivalent Bode diagram

Fourier frequency
Golden rule of locking: in open loop Bode diag., cross unitary gain with no more than 20dB/decade (unstable otherwise)
Phase-locking: PI corrector

Rule of thumb: \( \text{fint} < \text{fc}/3 \) to \( \text{fc}/10 \), otherwise unstable (too much residual dephasing at \( \text{fc} \) (unitary gain) due to integrator

**Open loop Bode diagram**

(remember \( \phi \rightarrow F = \text{pure integrator} \))

- **Noise sources**
- **RF/\( \mu \)-wave frequency to lock: F**
- **Phase detection**
- **Synthesizer**

**Physical system**

Vin \( \rightarrow \Delta F \)

**G [dB]**

- \( \text{fc} \) \( \rightarrow \) \( \text{unitary gain} \)
- \( \text{fint} \) \( \rightarrow \) \( 20 \text{dB/dec} \)
- \( \text{fint} \) \( \rightarrow \) \( 40 \text{dB/dec} \)
- \( \text{fc} \) \( \rightarrow \) \( 40 \text{dB/dec} \) from \( \text{fint} \) to \( \text{DC} \)

**PSD**

- \( \text{fint} \) \( \rightarrow \) \( \text{noise power spectral density} \)

**Reduction of noise**

- \( 20 \text{dB/dec} \) from \( \text{fc} \) to \( \text{fint} \)
- \( 40 \text{dB/dec} \) from \( \text{fint} \) to \( \text{DC} \)
Phase-locking: PI2 corrector

Rule of thumb: $f_{int2} < f_{int}/3$ to $f_{int}/10$, otherwise unstable

**Open loop Bode diagram**
- **Noise sources**
- **Physical system**: $Vin \rightarrow \Delta F$
- **RF/µ-wave frequency to lock**: $F$
- **Phase detection**
- **Synthesizer**

**PSD**
- Reduction of noise is 20 dB/dec from $fc$ to $f_{int}$
- 40 dB/dec from $f_{int}$ to $f_{int2}$
- 60 dB/dec from $f_{int2}$ to DC

**Noise power spectral density**

- **Unitary gain**: Keep weird stuff below unitary gain or you will be unstable
- **20 dB/dec**: $f_{int2}$
- **$f_{int}$**: $20 dB/dec$ from $fc$

** Fourier frequency**
Phase-locking: why you may need a divider

When loop is on:
- If $\sqrt{\int \text{PSD}} \ll 2\pi \implies \text{OK}$
(small excursions around 0)

(Mixer as a phase detector)

Point of lock
Phase-locking: why you may need a divider

When loop is on:
- If $\sqrt{\int \text{PSD}} < 2\pi$ → OK
- If not → cycle slips (BAD LOCK)

Note: mathematically, cycle-slips always happen, but maybe only once in a few months (negligible impact) if you are good enough…
Phase-locking: adding a divider

Exemple: divide by 4 (residual noise induces cycle slips if no division, not when dividing)
Phase-locking: digital systems

**Digital Phase detector (simple)**

With appropriate offset, locks at quadrature; same cycle slip issues than DBM.
Needs an analog to digital front-end (with good SNR)!

**Digital Phase-Frequency detector**

Same cycle slip issues than DBM, BUT natural auto-relock with large capture range.
Many different implementations exist, to address technical issues (blind zone at center, limited operation frequency,...)
Needs an analog to digital front-end (with good SNR)!
FPGA-based phase tracker with unwrap

Continuous phase tracking (limited only in BW by sampling rate of unwrapping) with potentially very large dynamics (limited only by number of bits per sample at unwrap output)

Limited in lock BW by delay in FPGA (typ. >400kHz with Redpitaya, )

https://github.com/jddes/Frequency-comb-DPLL
Phase-locking: application to frequency comb

1) Frequency multiplier scheme:

\[ f_b = \nu_l - (nf_{\text{rep}} + f_0) \]

\[ \nu_n = nf_{\text{rep}} + f_0 \]

Pros: simple, robust

Cons: noise of ref. is multiplied to optics domain

Laser to measure \( \nu_l \)

Pump power (acts on \( f_0 \), and a bit on \( \text{frep} \) too)

PZT (acts mainly on frep)

Optical frequency comb

Absolute reference (Maser, GPS, Cs clock, fountain, …)

Pros:

Cons:
Phase-locking: application to frequency comb

1) Frequency divider scheme:

\[ f_b = v_l - (n f_{rep} + f_0) \]

Pros:
- noise is divided
- low linewidth regime

Cons:
- PLL more difficult

\[ v_n = n f_{rep} + f_0 \]

Laser to measure

Pump power (acts on \( f_0 \), and a bit on \( f_{rep} \) too)

PZT (acts mainly on \( f_{rep} \))

Pros:
- noise is divided
- low linewidth regime

Cons:
- PLL more difficult

Absolute reference (Maser, GPS, Cs clock, fountain, …)

Synthesizer

Counter

Loop filter

Rep. rate detection
Phase-locking: application to frequency comb

1) Frequency divider scheme:

\[ f_b = v_l - \left( n f_{rep} + f_0 \right) \]

Pros:
- noise is divided
- low linewidth regime

Cons:
- PLL more difficult

Optical frequency comb

Absolute reference
(Maser, GPS, Cs clock, fountain, …)
Phase-locking: the low linewidth regime

In the frequency divider scheme:

- Better counting of 2\textsuperscript{nd} optical reference
- Narrow filter can save you if S/N is bad (you need $\sim$23dB S/N \textbf{in the measurement BW} to count or track; 10 dB in 100kHz is 30dB in 1kHz…)
- Allow direct spectroscopy of narrow atomic transition

Removing PLLs: the transfer oscillator technique

A clever use of arithmetics allows comparison of combs-related beatnotes where the free-running noise of the comb vanishes (can replace or be combined with PLLs)

1) Remove $f_0$ to make a virtual comb with $f_0=0$:

- Laser to measure $v_l$
- $v_n = nf_{rep} + f_0$
- $f_b = v_l - (nf_{rep} + f_0)$

1bis) One can also use an AOM, driven by $f_0$ to shift the comb to $f_0=0$ (much more expensive, but comb at $f_0=0$ is real instead of being virtual, which can be useful, for example in DFCS)


S. Koke et al., Nature Photonics 4, 462 (2010) (Steinmeyer) ; also at UWA (Luiten)
Removing PLLs: the transfer oscillator technique

A clever use of arithmetics allows comparison of comb-related beatnotes where the free-running noise of the comb vanishes.


2) Use arithmetics for removing $f_{\text{rep}}$:

Example 1: two optical frequencies to compare (frequency ratio of two clocks)

\[
\begin{align*}
\Delta f_{b1} - f_0 &= \nu_{l1} - n_1 f_{\text{rep}}(t) \\
\Delta f_{b2} - f_0 &= \nu_{l2} - n_2 f_{\text{rep}}(t)
\end{align*}
\]

\[
\begin{align*}
n_2 \cdot (\Delta f_{b1} - f_0) - n_1 \cdot (\Delta f_{b2} - f_0) &= n_1 \cdot \nu_{l1} - n_2 \cdot \nu_{l2}
\end{align*}
\]

Comparison between $\nu_{l1}$ and $\nu_{l2}$ is independant of the noise of the comb!

One can realize the arithmetic operation in hardware (with 1 or 2 DDS, which output frequency is really $f_{\text{out}} = f_{\text{clock-in}} \times (\text{ctrl word})/2^{\#\text{digits}}$) or directly in software from counters’ data, BUT one needs very well synchronised counter channels (otherwise, $f_{\text{rep}}(t)$ and $f_{\text{rep}}(t+\Delta t)$ don’t cancel each other out…). Also be aware of $\#\text{digits}$ in calculation: double floats in C ($\sim 10^{-15}$) are not enough unless you apply some tricks…
Removing PLLs: the transfer oscillator technique

Exemple 2: compare optical and microwave frequencies


\[ v_C = v_A - v_B = m_2 f_{LO} - \left( m_1 m_2 f_{rep} + \frac{v_{CEO} + \Delta x}{m_3} \right) \]

\[ m_1 m_2 f_{rep} + \frac{v_{CEO} + \Delta x}{m_3} = \frac{1}{m_3} v_x \]

\[ v_C = m_2 f_{LO} - \frac{v_x}{m_3} \]

The difference in order of magnitude between optics and RF domain is much larger than in the optics-optics case, therefore more advanced technique needs to be used for realizing the arithmetics…
<table>
<thead>
<tr>
<th>Pros</th>
<th>transfer oscillator</th>
<th>PLL</th>
</tr>
</thead>
<tbody>
<tr>
<td>No lock therefore no unlock</td>
<td></td>
<td>Real comb and not virtual</td>
</tr>
<tr>
<td>No lock therefore no lock-related cycle-slips</td>
<td></td>
<td>If narrow linewidth real combs, DFCS</td>
</tr>
<tr>
<td>Virtual comb is extremely good even though real comb is noisy</td>
<td></td>
<td>Easy to add a new wavelength (1 simple beatnote)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Cons</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Usually still requires a (loose) lock to stay inside filters</td>
<td></td>
<td>High bandwidth, low noise, high dynamics PLLs are not trivial</td>
</tr>
<tr>
<td>Only a virtual comb → direct spectroscopy is not feasible at low linewidth</td>
<td></td>
<td>Sometimes, the comb may go unlock (automatic relock may help)</td>
</tr>
<tr>
<td>Adding a new wavelength is a bit more complicated (RF circuit, DDS,…)</td>
<td></td>
<td>Perf. limited by how good you are at making PLLs</td>
</tr>
</tbody>
</table>

Note: you can also do both, or a bit of both (ex.: remove \( f_0 \) only)…
Nasty “details” about the first digits
Nasty “details” of absolute frequency measurements

The comb gives you an simple arithmetic relation between various frequencies (all measured frequencies are >0 by definition)

BUT

Some parameters (signs, multiplying integers) are not known a priori and need to be determined (with a bit of intelligence)

Exemple :

A comb measuring an optical frequency $\nu_{cw}$ against absolute RF references by counting (or locking to a fixed frequency) $f_{\text{rep}}$, $f_0$, $f_b$:

$$f_b = |\nu_l - \nu_N| = \pm (\nu_{cw} - N \cdot f_{\text{rep}} \pm f_0)$$

$$\nu_l = N \cdot f_{\text{rep}} \pm f_0 \pm f_b$$

Have to be determined
How to determine signs

Assume:

\[ \nu_1 = N \times f_{\text{rep}} \pm f_0 \pm f_b \]

With \( f_0 \) and \( f_b \) phase-locked to \( f_0^{\text{ref}} \) and \( f_b^{\text{ref}} \) by feed-back to the comb, while measuring \( f_{\text{rep}} \) with a counter (i.e. « frequency divider scheme »)

THEN

IF (\( f_0^{\text{ref}} \uparrow \) makes \( f_{\text{rep}} \downarrow \)) THEN the formula is with \( +f_0 \)

IF (\( f_0^{\text{ref}} \downarrow \) makes \( f_{\text{rep}} \uparrow \)) THEN the formula is with \( -f_0 \)

IF (\( f_b^{\text{ref}} \uparrow \) makes \( f_{\text{rep}} \downarrow \)) THEN the formula is with \( +f_b \)

IF (\( f_b^{\text{ref}} \downarrow \) makes \( f_{\text{rep}} \uparrow \)) THEN the formula is with \( +f_0 \)
How to determine $N$

Assumes the signs are determined to be:

- **Method 1:**
  
  If $v_1$ is known a priori with an accuracy better than $f_{\text{rep}}$, $f_0$ and $f_b$ (ex.: you have just measured it with a very good wavemeter, or it’s coming from a well known atomic/molecular reference which someone has measured previously for you, like Sr or Ca+ or…)
  
  $\Rightarrow N = \text{round to nearest integer } \left[ (v_1^{\text{a priori}} - f_b - f_0) / f_{\text{rep}} \right]$

- **Method 2:**
  
  Similar to sign determination: by shifting $f_{b\text{ref}}$:
  
  $f_{b\text{ref}} \rightarrow f_{b\text{ref}} + \Delta f_{b\text{ref}}$
  
  one induces:
  
  $f_{\text{rep}} \rightarrow f_{\text{rep}} + \Delta f_{\text{rep}}$

  $v_1 = N f_{\text{rep}} + f_0 + f_{\text{ref}} = N f_{\text{rep}} + \Delta f_{\text{rep}} + f_0 + (f_{b\text{ref}} + \Delta f_{b\text{ref}})$

  $\Rightarrow N = \text{round to nearest integer } \left[ -\Delta f_{b\text{ref}} / \Delta f_{\text{rep}} \right]$

⚠️ when noise of measurement is too high (and/or drift too fast inbetween measurements…)
this is (generally) not sufficient
ex.: $f_{\text{rep}} \sim 250 \text{ MHz measured at } \pm 100 \mu \text{Hz } (4 \times 10^{-13})$, $v_1 \sim 250 \text{THz}$ (hence, $N \sim 10^6$),
$\Delta f_{b\text{ref}} = 1 \text{MHz induces } \Delta f_{\text{rep}} \sim 1 \text{Hz}$
$\Rightarrow \Delta N / N \sim \sqrt{2.100 \mu \text{Hz} / \Delta f_{\text{rep}}} \sim 140 \times 10^{-6}$
$\Delta N \gg 1 \Rightarrow$ it DOESN’T WORK !!! (here, we would need $\sim 10^{-15}$ for $\Delta N < 1$)
How to determine N

Assumes the signs are determined to be:

\[ \nu_l = N \times f_{\text{rep}} + f_0 + f_b \]

- **Method 3:**

Make 2 measurements for 2 consecutive N (easy by looking at the beat line moving)

\[ \nu_l = N \times f_{\text{rep},1} + f_0 + f_b = (N+1) \times f_{\text{rep},2} + f_0 + f_b \]

\[ \Rightarrow N = \frac{f_{\text{rep},2}}{f_{\text{rep},1} - f_{\text{rep},2}} \]

Required measurement stability:

ex.: \( f_{\text{rep}} \approx 250 \text{ MHz measured at } \pm 100\mu\text{Hz} \) (4x10^{-13}), \( \nu_l \approx 250\text{THz} \) (hence, \( N \approx 10^6 \)),

\[ \Rightarrow |f_{\text{rep},1} - f_{\text{rep},2}| \approx 250\text{Hz} \]

\[ \Rightarrow \Delta N/N \approx 100\mu\text{Hz}/250\text{MHz} + \sqrt{2}. 100\mu\text{Hz}/250\text{Hz} \approx 6x10^{-7} \]

\[ \Delta N < 1 \Rightarrow \text{it WORKS (without crazy requirements !!!)} \]

(note for a few extra margin, one can increase the difference in N)

- **Method 4:** If you are really rich and have 2 combs, you can use method 2 or 3 with the 2 measurements done simultaneously with each comb...
A word about counting frequencies

- **Π-counter vs. Λ-counters:**
  - In Λ-counters, an averaging process smoothes the data to reduce the noise → the bandwidth of the measurement is extremely reduced!
  - With Π-counters, you should give the measurement BW of your data acquisition for meaningful comparisons.
  - Juxtaposing 1s-gate time data from a Λ-counter (like an HP53131 in gate time mode for example) DOESN’T give you an Allan variance !!!
  - It doesn’t give you exactly a MVAR either !

- Dead-time vs. no-dead-time counters: beware of minimizing dead-times (compared to gate times) or, better, find a counter without dead-times (ex. K+K messtechnik)

- Be careful when comparing numbers measured with one counter to another

- The only truth is in the power spectral density of (frequency or phase) noise; counting and AVAR-like calculation only gives easily access to an estimation (more or less fair) of the PSD at a given Fourier frequency

→ See S. Dawkins et al. ”Considerations on the Measurement of the Stability of Oscillators with Frequency Counters”, IEEE UFFC 54, 918 (2007) for more details
Other Applications of Optical Frequency Combs

- **Ultra low noise microwave generation with fiber femtosecond laser**
  
  *The Stability of an Optical Clock Laser Transferred to the Interrogation Oscillator for a Cs Fountain*
  
  *Ultralow noise microwave generation with fiber-based optical frequency comb and application to atomic fountain clock*, J. Millo et al., Appl. Phys. Lett. 94, 141105 (2009)

  
  X.Xie, R. Bouchand et al., Nat. Phot. 11, 44 (2017) → **Lowest noise microwave source**

- **Transfer of spectral purity from near-IR to other wavelengths**
  
  D. Nicolodi et al., Nat. Phot. 8, 219 (2014) (NIR)
  
  B. Argence et al., Nat. Phot. 9, 456 (2015) (MIR)

- **Astrocombs: Use of OFC as reference for high resolution spectrometers used in astronomical telescopes**
  
  
  
  Science 321, 1335 (2008.)

- **Direct Frequency Comb Spectroscopy, CEAS, dual combs spectroscopy**
  
  *Direct frequency comb spectroscopy*,
  

- **Molecular fingerprinting**
  
  *Molecular fingerprinting with the resolved modes of a femtosecond laser frequency comb*,
  

- **Extension to the DUV/XUV range**
  
  *A deep-UV optical frequency comb at 205 nm*
  
  Peters, E. et al., Optics Express 17, 9183 (2009)
  
  *Efficient output coupling of intracavity high-harmonic generation*
  
A few extra goodies
Transfer of optical spectral purity to the microwave domain (Low phase noise microwave signal generation)
Low-noise μwave: motivation

Existing very low f-noise μ-wave sources (~10GHz):
- Room temp Sapphire osc. (Raytheon, formerly Poseidon Australia):
  -40dBC/Hz @ 1Hz, -170dBC/Hz @ 100kHz from carrier
- Cryogenic Sapphire oscillator (UWA, FEMTO-ST, ULISS):
  -100dBC/Hz @1Hz, -140dBC/Hz @ 100kHz from carrier
- Opto Electronic Oscillator (JPL/OEwaves):
  -40dBC/Hz@1Hz, -160dBC/Hz @ 10kHz (large resonances after that)

Applications:
- atomic frequency standards
- radar
- VLBI
- synchronization of particle accelerators
- time reference distribution
- telecommunication
- …
Frequency division, effect on phase noise

fc [Hz] → fc/N [Hz] then $\Delta\phi \rightarrow \Delta\phi/N$ [rad]

$S_{\phi}(f)$ [dBc/Hz] → $S_{\phi}(f)-20\log_{10}(N)$ [dBc/Hz]

Large noise reduction if N is large…
USL transferred to μ-wave (projection)

A robust 4.5x10⁻¹⁶ (@1s) level USL cavity (designed following space industry standards and methods) → 10cm long cavity with rings

Prototype designed for transport +/-10g and operation at zero-2g
Currently existing lab prototype

Φ-noise of a 10 GHz carrier obtained by frequency division of the space-prototype USL at 200THz (SODERN/CNES/SYRTE), by a frequency comb, assuming perfect division.

200 THz (λ=1.5μm)

Φ-noise → -20.log(20000) = -86dB (!!!)

Low noise μ-wave generation with comb (optical frequency divider scheme)

\[ f_0 \]

\[ f_b = \nu_l - (Nf_{\text{rep}} + f_0) \]

\[ \nu_l \sim 200 \text{THz} \]

\[ \nu_n = Nf_{\text{rep}} + f_0 \]

\[ f_0 \text{ not locked } \rightarrow \text{virtual comb with } f_0 = 0 \]

Increase SNR for lower white phase noise floor

Thermal noise (Johnson-Nyquist):
A 0 dBm µ-wave signal cannot have a white phase noise limit better than -177dBc/Hz
Solution: increase µ-wave power
- higher optical power + more linear PD
(in coll. with Discovery semiconductor)
- high rep rate fs laser / external rep rate multiplication
  → less power in undesired harmonics, more in the harmonic of interest

Differential phase noise (common USL)

2 combs 2 MZM 2 PD (excess phase noise to investigate)
1 combs 2 MZM 2 PD

Optics Letters 36, 3654 (2011)
AMPM conv. in $f_{\text{rep}}$ and harmonics photodetection

amplitude fluctuations of the fs laser induce fluctuations of phase of $f_{\text{rep}}$
(and its harmonics)

→ possible to lock amplitude (but only at low Fourier frequencies)

1.2x10^{-16}@1s generated μwave / 100as synchro

→ or analyze carefully the physics...

By space-charge screening effect, close to saturation, the PD response is asymmetric

→ AM noise produces PM noise

For harmonic order >1 there are special situations...

Suppression of AMPM conversion

Reasonable to passively keep the laser power at ~0.1% 
→ From slope of EPC vs. Energy per pulse, $EPC < 0.003$ rad 
→ AMPM-induced excess phase noise $= RIN - 50dB$ !!!

Note: EPC coeff. is really alternatively $>0$ and $<0$, only $|EPC|$ is plotted here...

In this area $\rightarrow$ low EPC

Applied Physics B 106, 301 (2012)
Optics Letters 39, 1204 (2014)
Latest results for low phase-noise microwave

This is the lowest phase noise microwave source of all existing technologies and for ~all Fourier frequencies!


→ This is the lowest phase noise microwave source of all existing technologies and for ~all Fourier frequencies!
Transfer of spectral purity in the optics domain
Motivation

Toward a new paradigm in the optical clock community...

Now:

Ultra-stable laser


Other clocks + USLO (optical, µwave,...)

Optical clock (probing atomic/molecular transition)
Motivation

Toward a new paradigm in the optical clock community...

Now:

Ultra-stable laser


Other clocks + USLO (optical, μwave, …)

Optical clock (probing atomic/molecular transition)

Xfer of spectral purity.

Future:

Next generation USL (very difficult to make, expensive and probably wavelength specific)

Clock 1 (atom 1)

Clock 2 (atom 2) etc…
Transfer of spectral purity at the $10^{-18}$ (@1s) level

- Don’t rely entirely on the lock of the comb to 1.5µm to guess the phase of the combs’ teeth wrt to 1.5µm laser (especially those seen after EDFA+HNLF)
- All spectral phase fluctuations of EDFA+HNLF+comb scaling with $\nu_{\text{optic}}$ up to linear order are common-mode if both 1µm and 1.5µm are measured in the beat detection unit (BUT, low SNR…)

For test/characterization purpose

(or similar system)
Test of spectral purity transfer
Transfer of spectral purity at the $10^{-18}$ (@1s) level

Limitation at a few $10^{-18}$ at ~1s
(>1 order of magnitude lower than current best demonstrated USL)

Transfer of spectral purity in the MIR domain
THz motivation

Spectroscopy of molecules at high resolution:
- more complex, richer structure than atoms
- can be more sensitive to certain effects
- fundamental tests of physics
  (EDM, time-space variation, of fundamental constants, ...)

Exemple : test of parity violation in molecules
- due to weak interaction
  - seen in atoms and nuclear physics
  - never seen in molecule because too weak
- ultra-high resolution spectroscopy in chiral molecules ($\Delta \nu / \nu < 10^{-13}$)

Mid-IR QCL ideal tool for ro-vibrational lines (tunable, relatively easy to operate, ...)
BUT large free-running linewidths (~1MHz)
→ necessary to servo its frequency
QCL stabilization onto a NIR frequency reference

**SYRTE**
- Primary clocks
  - accuracy
- Ultrastable 1.54 µm laser
  - stability

**LPL**
- Optical frequency comb
  - Spatial transfer
- QCL
  - Spectral transfer
- High resolution molecular spectroscopy

Near-IR frequency reference
- 1s-stability \( \sim 10^{-15} \)
- Accuracy \( \sim 10^{-14} \)
  (with primary clocks: \( 3 \times 10^{-16} \))

Optical fiber link
- Free running stability \( \sim \leq 10^{-14} \)
- Stabilized: \( 10^{-15} \tau^{-1} \) from 1 to \( 10^4 \) s
QCL frequency stabilization

Δ₀ = ν_{ref} - N f_{rep}

Δ₁ = ν_{QCL} - (p-q) f_{rep}

ν_{QCL} = n f_{rep} + Δ₁

ν_{QCL} \sim \frac{n}{N} ν_{ref}

Sum-frequency generation with 9-11 µm tunability
QCL performance evaluation

MIR reference

CO$_2$ laser
10 µm

Servo loop

Saturated absorption line

OsO$_4$

QCL

1.82 µm

1.55 µm

OFC

AgGaSe$_2$

10.3 µm

PLL

Δ$_1$

Δ$_2$

MIR reference/QCL beatnote

PD1

PD2
Locked QCL performance

\[ \Rightarrow \text{QCL frequency noise } \sim 10^{-2} \text{ Hz}^2/\text{Hz between 1 and 100 Hz} \]

\[ \Rightarrow \text{QCL linewidth } \sim 0.2 \text{ Hz (7x10}^{-15} \text{)} \]

\[ \Rightarrow \text{QCL stability } \sim 1.5 \times 10^{-15} \text{ (1s-100s)} \]